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Cylindrical configurations of classical string with rigidity

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Abstract. All possible cylindrical configurations of classical Nambu–Goto–Polyakov string theory are obtained. Their cross sections are shown to be different lines which can be classified into eight different types: straight, undulating, positively self-intersecting, eight-like, negatively self-intersecting, buckling, nodoid-like and circular. These lines have the same shapes as elastic thin rods or strings which were known a century ago.

Classical string [1] with rigidity has attracted much attention [2–6] since the pioneering work of Polyakov [2]. A very recent attempt at such a kind of string theory with a brief review can be found in [6]. However, few results concerning its classical configurations (worldsheets) and analysis of their properties have been obtained [5]. In this paper, we give all possible classical cylindrical configurations in the standard Nambu–Goto–Polyakov string, which is the simplest classical string with a rigidity. By cylindrical configuration, we mean that the configuration is independent of the Z axis and the regular circular cylinder with circular cross section in the X–Y plane is only a special case of the configurations.

The simplest classical string with a rigidity, or the standard Nambu–Goto–Polyakov string, takes the action [1, 2, 5]

$$S = \frac{1}{\alpha_0}\int H^2\,\mathrm{d}A + \mu_0\int \mathrm{d}A \tag{1}$$

where $\mathrm{d}A$ is the area element, H is the mean curvature, α_0 and μ_0 are two coupling constants. The corresponding Euler–Lagrange equation is [5, 7, 8]

$$\nabla^2 H + 2H(H^2 - K) - 2\alpha_0\mu_0 H = 0 \tag{2}$$

where $\nabla^2 = \frac{1}{\sqrt{g}}\partial_i(g^{ij}\sqrt{g}\partial_j)$ is the Laplace–Beltrami operator, K is the Gaussian curvature. The physical parameter is the product of two constants, $\mu_0\alpha_0$, which is also called a coupling constant and is denoted by C, which may be positive, negative or zero [1, 2].

Since we limit ourself to the cylindrical case, we have zero-Gaussian curvature $K = 0$, and the general parametric form describing the cylindrical configuration is

$$\boldsymbol{R} = (X(s), Y(s), Z) \qquad -\infty < s < \infty \qquad -\infty < Z < \infty. \tag{3}$$

The first and second fundamental forms are

$$\begin{aligned} I &= \mathrm{d}s^2 + \mathrm{d}Z^2 \\ II &= (X''Y' - Y''X')\,\mathrm{d}s^2 \end{aligned} \tag{4}$$

where the prime ‘′’ stands for the derivative with respect to parameter s. The second fundamental form can be simplified if we take the parameter s as the arc-length of the curve representing the cross section (in the X–Y plane) of the cylindrical worldsheet. Thus, we have

$$\frac{\mathrm{d}X}{\mathrm{d}s} = \cos\psi(s) \qquad \frac{\mathrm{d}Y}{\mathrm{d}s} = \sin\psi(s) \tag{5}$$

where $\psi(s)$ is the angle of the tangent measured from the X-axis. The mean curvature H is

$$H = -\frac{1}{2}\frac{\mathrm{d}\psi(s)}{\mathrm{d}s}. \tag{6}$$

This leads to a useful relation

$$\frac{\mathrm{d}H}{\mathrm{d}s} = \frac{\mathrm{d}H}{\mathrm{d}\psi}\frac{\mathrm{d}\psi}{\mathrm{d}s} = -\frac{\mathrm{d}H^2}{\mathrm{d}\psi}. \tag{7}$$

Only those configurations satisfying the Euler–Lagrange equation (2) are stable. In our cylindrical case, equation (2) becomes

$$\frac{\mathrm{d}^2H}{\mathrm{d}s^2} + 2H^3 - 2CH = 0. \tag{8}$$

The first integral gives

$$\left(\frac{\mathrm{d}H}{\mathrm{d}s}\right)^2 = 2CH^2 - H^4 + c_1 \tag{9}$$

where c_1 is an integral constant. From equation (6), we have

$$\frac{\mathrm{d}H^2}{\mathrm{d}\psi} = \pm(2CH^2 - H^4 + c_1)^{1/2}. \tag{10}$$

Before transforming the above equation into the following elementary integral

$$\frac{\mathrm{d}H^2}{(2CH^2 - H^4 + c_1)^{1/2}} = \pm\mathrm{d}\psi \tag{11}$$

we must check whether the roots of the following equation, $2CH^2 - H^4 + c_1 = 0$, are possible solutions of equation (8). It is easy to realize that there are two important solutions:

$$\psi(s) = c_{r1} \qquad \psi(s) = \pm 2\sqrt{C}s + c_{r2} \tag{12}$$

where c_{r1} and c_{r2} are two integral constants. These two solutions correspond to flat planes and regular circular cylinders, respectively. The radii of the circular cylinders are $1/(2\sqrt{C})(C > 0)$. Then we give the results of integral (11) as

$$H^2 = C + \sqrt{C^2 + c_1}\sin(\pm\psi + c_2) \tag{13}$$

where c_2 is another integral constant, but its choice is a matter of deciding the direction of the X-axis from which the angle ψ is measured; we can therefore put $c_2 = 0$. The same applies to the choice of the positive or negative sign before ψ. Thus, it suffices to study the following equation:

$$H^2 = C + \sqrt{C^2 + c_1}\sin\psi. \tag{14}$$

From equation (6) it becomes

$$\left(-\frac{1}{2}\frac{\mathrm{d}\psi(s)}{\mathrm{d}s}\right)^2 = C + \sqrt{C^2 + c_1}\sin\psi \tag{15}$$

i.e.,

$$\frac{\mathrm{d}\psi}{(C + \sqrt{C^2 + c_1}\sin\psi)^{1/2}} = \pm 2\,\mathrm{d}s. \tag{16}$$

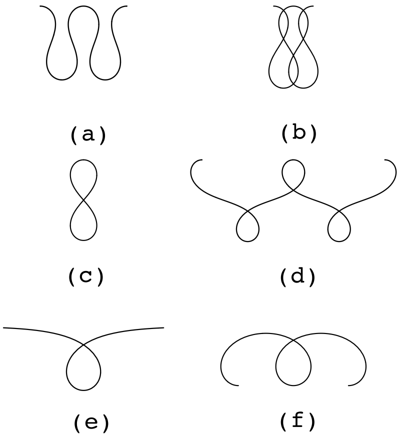

Figure 1. Cross sections of the possible classical configurations of string theory. All these curves, except (*c*) and (*e*), are periodic horizontally.

Detailing the sign of $\pm$ before ds is a matter of choosing the direction of the angle ψ, we therefore choose the positive sign for convenience. Multiplying both sides of the above equation by $\cos\psi$, we have

$$\frac{\mathrm{d}\sin\psi}{\left(C+\sqrt{C^2+c_1}\sin\psi\right)^{1/2}} = 2\,\mathrm{d}X \tag{17}$$

i.e.,

$$\sin\psi = \sqrt{C^2+c_1}(X-X_0)^2 - C \tag{18}$$

where X_0 is an integral constant which can be chosen as zero. This equation can give the cross sections of all possible classical cylindrical configurations except the regular circular cylinder. By using equation (5) in the following integral form

$$Y(X) - Y(X_0) = \int_{X_0}^{X} \tan\psi\,\mathrm{d}X \tag{19}$$

we can obtain all kinds of curves of the cross sections in the X–Y plane. In figure 1, we show six different typical types of these curves. Among these curves, the one shown by figure 1(*a*) is the only type which is not self-intersecting; and all curves except those shown by figures 1(*c*) and (*e*) can be viewed as basic units which are periodic horizontally, i.e. periodic in the Y-axis direction.

These curves appear very complicated. In order to obtain a quick understanding, it is helpful to look at these curves via the relation of tangent angle ψ in terms of the arc-length, instead of the X-axis coordinate. The direct integral of the differential equation (16) can lead to the following results:

$$\sin\psi = 1 - 2\,\mathrm{sn}^2\left[2\left(C+\sqrt{C^2+c_1}\right)^{1/2} s, k_1\right] \qquad \text{for } c_1 < 0 \text{ and } C > 0 \tag{20}$$

$$\tan\psi = \exp\left(\sqrt{2C}s\right) \qquad \text{for } c_1 = 0 \text{ and } C > 0 \tag{21}$$

$$\sin\psi = 1 - k_2^2\,\mathrm{sn}^2[\sqrt{2}(C^2+c_1)^{1/4}s, k_2] \qquad \text{for } c_1 > 0\text{, whenever } C > 0 \text{ or } C < 0 \tag{22}$$

where sn$[s, k]$ is the Jacobian elliptic sine function, and the two moduli k_1 and k_2 are

$$\left(\frac{2\sqrt{C^2+c_1}}{(C+\sqrt{C^2+c_1})}\right)^{1/2} \qquad \text{and} \qquad \left(\frac{C+\sqrt{C^2+c_1}}{2\sqrt{C^2+c_1}}\right)^{1/2}$$

respectively. Since the elliptic function is a periodic function, we can see that in case $c_1 \neq 0$, the curves are periodic. The curvatures $c = \mathrm{d}\psi/\mathrm{d}s$ are respectively given by

$$c = -2\left(C + \sqrt{C^2 + c_1}\right)^{1/2} \mathrm{dn}^2\left[2\left(C + \sqrt{C^2 + c_1}\right)^{1/2} s, k_1\right] \qquad \text{for } c_1 < 0 \text{ and } C > 0 \tag{23}$$

$$c = 2\sqrt{2C}\,\mathrm{sech}\,\sqrt{2C}s \qquad \text{for } c_1 = 0 \text{ and } C > 0 \tag{24}$$

$$c = -2k_2\sqrt{2}(C^2 + c_1)^{1/4}\,\mathrm{cn}^2[\sqrt{2}(C^2 + c_1)^{1/4}s, k_2] \quad \text{for } c_1 > 0 \text{ whenever } C > 0 \text{ or } C < 0. \tag{25}$$

The angles $\psi_0 = \int_0^T c\,\mathrm{d}s$ rotated in one period T, are accordingly:

$$\psi_0 = 2\pi \qquad \text{for } c_1 < 0 \text{ and } C > 0 \tag{26}$$

$$\psi_0 = 2\pi \qquad \text{for } c_1 = 0 \text{ and } C > 0 \tag{27}$$

$$\psi_0 = 0 \qquad \text{for } c_1 > 0 \text{ whenever } C > 0 \text{ or } C < 0. \tag{28}$$

Using this rotation angle ψ_0, we can easily understand these configurations. When $C < 0$ and $c_1 > 0$, the curves take undulation shapes as shown in figure 1(*a*). When $C > 0$ and $c_1 > 0$, with c_1 decreasing from a relatively large value, the curves not only take undulation shapes as shown in figure 1(*a*), but also take the positively self-intersecting shapes as shown in figure 1(*b*), eight-like shapes as shown in figure 1(*c*) and negatively self-intersecting shapes as shown in figure 1(*d*). Both figures 1(*b*) and (*d*) are self-intersecting, but figure 1(*b*) is positively self-intersecting and figure 1(*d*) is the negatively self-intersecting. When $c_1 = 0$, the period is infinity; and the curve takes a buckling shape as shown in figure 1(*e*). When $c_1 < 0$ and $C > 0$, the curve takes nodoid-like shapes as shown in figure 1(*f*). Besides, when the coupling constant $C > 0$, we can have the regular circular cylinder. Whenever the coupling constant is positive, negative or zero, the flat plane is always a possible configuration.

Finally, we would like to point out the fact that the results of this paper have some significance in two other seemingly distinct areas. First, all configurations in figure 1 offer an exhaustive representation of the types of shapes of elastic thin rods or strings, which have been known since the last century, with an entirely different treatment [9]. Second, if taking two constants $1/\alpha_0$ and μ_0 in equation (1) as the bending stiffness and the surface tension coefficients, respectively, the energy functional for biomembranes with the zero-spontaneous curvature and zero-osmotic pressure difference share exactly the same form as equation (1). The undulation shapes in figure 1 were observed in an egg lecithin swelling in excess water [10], as pointed out in [11].

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